Stress-Strain Relations for Materials with Different Moduli in Tension and Compression

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Models in the form of stress-strain, or constitutive, relations are discussed for materials with moduli under tensile loading which are different from those under compressive loading. Criteria for consistent material models are given which are based on the principles of anisotropic elasticity and on the known behavior of such materials. The Ambartsumyan material model is compared with the criteria and found to violate the requirement of symmetric compliances. An improved model, called the weighted compliance matrix (WCM) material model, is shown to satisfy the criteria for isotropic and orthotropic bodies under plane stress. The new model can be extended by deduction to more complicated situations such as anisotropic bodies under general stress states.

Nomenclature

E = Young's modulus
E_c = Young's modulus in compression (Fig. 1)
E_t = Young's modulus in tension (Fig. 1)
E^{45} = Young's modulus at 45° to principal material directions
G_{tc} = shearing modulus [Eq. (4)]
G_{mn} = shearing modulus in the principal material directions
k_p, k_q = weighting factors in compliance matrix [Eq. (13)]
S_{ij} = compliances in strain-stress relations [Eq. (2)]
ν = Poisson's ratio
ν_{mn} = Poisson's ratio for contraction (expansion) in the n direction due to extension (compression) in the m direction

Subscripts

m,n = principal material directions
c = compression
p,q = principal stress directions
t = tension
tc = abbreviation for tension or compression, as appropriate

Superscripts

pq = principal stress coordinates
mn = principal material coordinates
xy = arbitrary coordinates at angle ω from principal stress coordinates (Fig. 5)
45 = at 45° to principal material directions

Introduction

COMPOSITE materials are receiving increasing attention in structural applications because of important weight savings. The weight savings are a result of the combination of a light, weak, and flexible matrix material with a very strong and stiff reinforcing material in the form of fibers or granules. The resulting composite material is light, yet strong and stiff.

Fiber-reinforced composite materials are used in a wide variety of applications, ranging from laminated aircraft wings to golf club shafts. Granular composite materials are used in re-entry vehicle nose tips, nuclear reactor control rods, etc. Incidentally, with the current energy crisis, accurate stress analysis of reactor control rods is essential to an understanding of their behavior in order to prevent failures. Otherwise, further repetition of control rod failures with subsequent reactor shutdowns will prevent nuclear energy sources from attaining their obviously needed potential. Accurate stress analysis is no less essential in most other applications of composite materials.

One of the important characteristics of composite materials is that they often exhibit different moduli or stiffnesses under tensile loading than under compressive loading. This characteristic behavior is shown schematically in the stress-strain curve of Fig. 1. Both fiber-reinforced and granular composite materials have different moduli in tension and compression as displayed in Table 1. Unidirectional glass fibers in an epoxy matrix have compression moduli 20% lower than the tension moduli.[1] For some unidirectional boron/epoxy fiber-reinforced laminae, the compression moduli are about 15-20% larger than the tension moduli.[2] In contrast, some unidirectional graphite/epoxy fiber-reinforced laminae have tension moduli up to 40% greater than the compression moduli.[2] Other fiber-reinforced composites such as carbon/carbon have tension moduli from two to five times the compression moduli.[3] Thus, no clear pattern of larger tension than compression moduli or vice versa exists for fiber-reinforced composite materials. A plausible physical explanation for this puzzling circumstance has yet to be made. For granular composite materials, the picture is no clearer. ZTA graphite has tension moduli as much as 20% lower than the compression moduli.[4] On the other hand, ATJ-S graphite has tension moduli as much as 20% more than the compression moduli.[5]

Many materials have different tension and compression moduli. Which modulus is higher may depend on the fiber or granule stiffness relative to the matrix stiffness. Such a relationship would influence whether the fibers or granules tend to contact and thereby stiffen the composite. Or, the individual fibers could buckle, leading to a less stiff composite. A general physical explanation of the reasons for different behavior in tension and compression is not yet available. Investigation of the micromechanical behavioral aspects of composite materials may lead to a rational explanation of this phenomenon. Until such an explanation is available, the apparent behavior can be used in analyzing the stress-strain behavior of materials. Thus, even without knowing why the materials behave as they do, we can model their apparent behavior.

Actual stress-strain behavior is not as simple as shown in Fig. 1. Instead, a nonlinear transition region may exist be-

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Table 1 Tension and compression moduli relationships for several common composite materials

Material	Fibrous or granular	Representative moduli relationship
Glass/epoxy	Fibrous	$E_t = 1.25E_c$
Boron/epoxy	Fibrous	$E_c = 1.2E_t$
Graphite/epoxy	Fibrous	$E_t = 1.4E_c$
Carbon/carbon	Fibrous	$E_t = 2\text{-}5E_c$
ZTA graphite	Granular	$E_c = 1.2E_t$
ATJ-S graphite	Granular	$E_t = 1.2E_c$

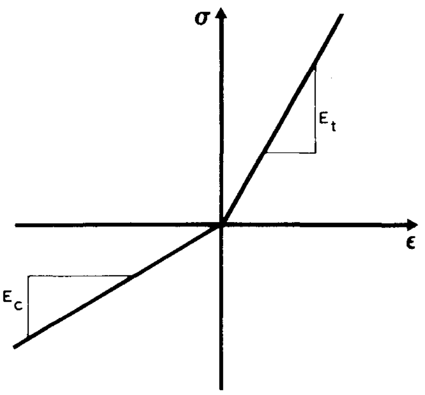

Fig. 1 Stress-strain curve for a material with different moduli in tension and compression.

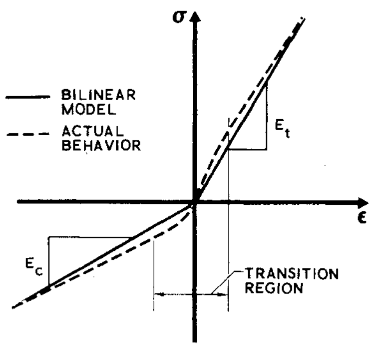

Fig. 2 Comparison of actual stress-strain behavior with the bilinear model.

tween the tension and compression linear portions of the stress-strain curve.[6] The measurement of strains near zero stress is difficult to perform accurately, but the stress-strain behavior might be as shown in Fig. 2, where replacement of the actual behavior by the bilinear model is offered as a simplification of the obviously nonlinear behavior. For most materials, the mechanical property data are insufficient to justify use of a more complex material model. One possible disadvantage of the bilinear stress-strain curve approximation is that a discontinuity in slope (modulus) occurs at the origin of the stress-strain curve.

Given that the uniaxial stress-strain behavior is approximated by a bilinear representation, the definition remains of the actual multiaxial stress-strain, or constitutive, relations that are required in structural analysis. Over the past 10 years, Ambartsumyan and his co-workers,[7-11] in the process of obtaining solutions for stress analysis of shells and bodies of revolution, defined a set of stress-strain relations which will be referred to herein as the Ambartsumyan material model.† Tabaddor[12] elaborated somewhat on the Ambartsumyan material model. Jones[13] applied the model to the problem of buckling under biaxial loading of circular cylindrical shells made of an isotropic material. However, in application of the Ambartsumyan material model to orthotropic materials, certain characteristics, such as a nonsymmetric compliance matrix in the stress-strain relations,[11] are apparent.

In the present paper, criteria are established for a consistent material model. The criteria are based on the concepts of anisotropic elasticity and on the known behavior of materials with different tension and compression moduli. An improved material model, in the form of consistent stress-strain relations, is then defined. Both the Ambartsumyan material model and the author's improved material model, called the weighted compliance matrix (WCM) model, are illustrated for isotropic and orthotropic materials under plane stress. Other more general stress cases and more general materials, such as anisotropic materials, can be treated by deduction.

†The reader is cautioned that the Russian word "soprotivlenie" often is translated incorrectly in the context of the present topic as "strength" rather than its proper meaning of "stiffness" or "resistance." This situation is quite unfortunate for those individuals who must select key words carefully in literature searches in order to avoid being overwhelmed by information; i.e., they are very likely to miss papers on the present topic.

Criteria for Consistent Material Model

A material model is defined by its stress-strain relations or, alternatively, strain-stress relations. For example, consider the strain-stress relations in principal stress (p-q) coordinates for an orthotropic material under a condition of plane stress

$$\epsilon_p = S_{11}^{pq}\sigma_p + S_{12}^{pq}\sigma_q \tag{1a}$$

$$\epsilon_q = S_{21}^{pq}\sigma_p + S_{22}^{pq}\sigma_q \tag{1b}$$

$$\gamma_{pq} = S_{61}^{pq}\sigma_p + S_{62}^{pq}\sigma_q \tag{1c}$$

(Notation for compliances of three-dimensional stress-strain relations, i.e., subscripts from 1 to 6, is used in accordance with conventional composite materials terminology.[14]) Note that, because of the nonalignment of principal stress (p-q) coordinates with principal material (m-n) coordinates, γ_{pq} is not zero. That is, principal strain directions generally do not coincide with principal stress directions for an orthotropic material. The choice of the values of the compliances, S_{ij}^{pq}, in Eq. (1) [or, alternatively, the stiffnesses C_{ij} in the inverse of Eq. (1), the stress-strain relations] constitutes the definition of the mathematical model for the material under consideration. We will find that an equivalent *orthotropic* material will be defined for isotropic materials with different moduli in tension and compression. Moreover, an equivalent *anisotropic* material will be defined for multimodulus orthotropic materials. (The term "multimodulus" is used as an abbreviation for different moduli in tension and compression.)

A material model must satisfy certain criteria in order to be called consistent. These criteria are based on the concepts of anisotropic elasticity and on the known behavior of materials with different tension and compression moduli. Specifically, the strain-stress relations must satisfy the following criteria:

1) The compliances S_{ij} and the moduli (stiffnesses) must be symmetric in principal material coordinates and in all other coordinate systems in order for the strain energy to be positive definite (so that energy is not created under deformation), and therefore a potential function exists.[15]

2) The values of the compliances are restricted in relation to one another such that the compliance matrix is positive definite.[15]

3) The value of the compliances must depend on the multiaxial principal stress state.

4) The shear modulus for an orthotropic material with different moduli in tension and compression must have different values upon imposition of shear stresses of opposite sign in coordinates other than the principal material directions.

5) The compliances and moduli for multimodulus orthotropic materials must reduce properly to those for multimodulus isotropic materials upon equating properties in

principal material directions. Moreover, the compliances and moduli for multimodulus isotropic materials must reduce to those for ordinary isotropic materials upon equating properties in tension and compression.

The imposition of symmetry on the compliance matrix (criterion 1) has an obvious impact on the form of the material property constants (although this condition is not met easily in a rigorous manner). However, the associated positive-definiteness of the strain energy (criterion 2) is shown by Lempriere[15] to lead to several nonobvious conditions on the acceptable Poisson's ratios. Moreover, criteria 1 and 2 lead to the conclusion that not all tension and compression properties for multimodulus materials are independent. This conclusion follows from the observation that the contraction in the y-direction due to a tensile load in the x-direction must be affected by the compressive stiffness in the y-direction. That is, the directional interaction, or cross-compliance, terms must in some manner be dependent even though the direct terms such as moduli are independent.

The *multiaxial* stress state in criterion 3 is essential for description of the basic problem of mixed states of tensile and compressive stress. Moreover, the merit of using principal stresses is that the strain-stress relations are simpler in principal stress coordinates than in any other coordinates. Principal stresses have a special meaning for isotropic materials, especially in failure calculations, since no principal material directions exist. However, for orthotropic materials, principal material directions are all-important, and principal stress directions ordinarily have no significance. (They are, in fact, immaterial in failure calculations.) Our use of principal stresses is a unique exception, as we will understand later.

Criterion 4, about different shear moduli when the shear stress is reversed, can best be explained with the aid of sketches of shear stresses and their associated principal stresses. When a positive or negative shear stress is applied in principal material directions to a unidirectionally reinforced lamina, the corresponding principal stress states are mirror images of one another as in Fig. 3. Hence, the apparent shear modulus is independent of shear stress sign. However, if a positive or negative shear stress is applied at any other angle, say 45°, to the principal material directions, then the corresponding principal stresses are not alike. The normal stresses in Fig. 4 are of opposite signs on the fibers. There, for positive shear stress, tensile stresses result in the fiber direction, and compressive stresses arise perpendicular to the fibers. For negative shear stress, compressive stresses exist in the fiber direction and tensile stresses transverse to the fibers. However, the moduli are different in tension than in compression. Thus, the deformations and hence the apparent shear moduli are different under positive and negative shear stress in other than principal material directions.

Criterion 5, about reduction to simpler classes of materials, is an obvious requirement for all multimodulus models. This criterion is what leads to the consistent use of principal stress coordinates for both isotropic and orthotropic multimodulus models. If principal material directions were used for an orthotropic multimodulus model, then that model would not reduce to an isotropic model because principal material directions do not exist for isotropic materials even if they are multimodulus. Thus, the only alternative is used: principal stress directions that are defined for both materials.

In the sections that follow, the Ambartsumyan material model will be compared with the preceding criteria and found to be unsatisfactory. An improved material model for which the criteria are satisfied then is suggested. Finally, the two material models are compared for a realistic material under a simple state of plane stress.

Ambartsumyan Material Model

Ambartsumyan[7] developed a set of stress-strain relations for isotropic materials with different moduli in tension and compression. Those stress-strain relations will be referred to

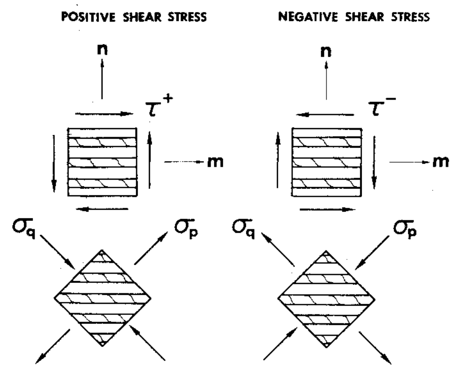

Fig. 3 Positive and negative shear stress in principal material directions.

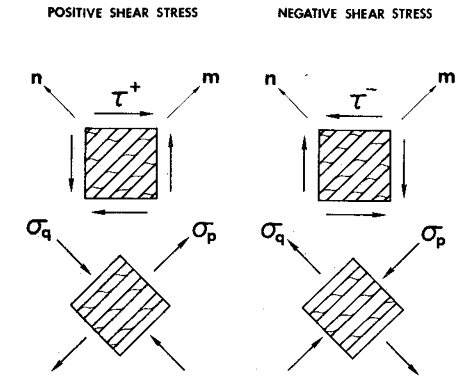

Fig. 4 Positive and negative shear stress at 45° to principal material directions.

as the Ambartsumyan material model. Over the past 10 years, Ambartsumyan and his associates further developed the model and applied it to elasticity and shell problems.[7-11]

For an isotropic body under plane stress, the Ambartsumyan strain-stress relations in principal stress coordinates are

$$\epsilon_p = S_{11}^{pq}\sigma_p + S_{12}^{pq}\sigma_q, \quad \epsilon_q = S_{21}^{pq}\sigma_p + S_{22}^{pq}\sigma_q \tag{2}$$

where the compliances S_{ij}^{pq} are related to the technical constants (the elastic moduli and Poisson's ratios) as follows:

if $\sigma_p > 0$ and $\sigma_q > 0$:

$$S_{11}^{pq} = 1/E_t, S_{12}^{pq} = S_{21}^{pq} = -\nu_t/E_t, S_{22}^{pq} = 1/E_t \tag{3a}$$

if $\sigma_p < 0$ and $\sigma_q < 0$:

$$S_{11}^{pq} = 1/E_c, S_{12}^{pq} = S_{21}^{pq} = -\nu_c/E_c, S_{22}^{pq} = 1/E_c \tag{3b}$$

if $\sigma_p > 0$ and $\sigma_q < 0$:

$$S_{11}^{pq} = 1/E_t, S_{12}^{pq} = -\nu_c/E_c, S_{21}^{pq} = -\nu_t/E_t, S_{22}^{pq} = 1/E_c \tag{3c}$$

if $\sigma_p < 0$ and $\sigma_q > 0$:

$$S_{11}^{pq} = 1/E_c, S_{12}^{pq} = -\nu_t/E_t, S_{21}^{pq} = -\nu_c/E_c, S_{22}^{pq} = 1/E_t \tag{3d}$$

Thus, an isotropic multimodulus material is effectively an orthotropic material.

The strain-stress relations in Eq. (2) can easily be inverted to obtain stress-strain relations. The relations can be rotated to nonprincipal stress coordinates and in that case must be supplemented by a shear stress - shear strain relation wherein the shear modulus is

$$G_{tc} = \left\{2\left[(S_{11}^{pq} - S_{12}^{pq})\frac{\sigma_p}{\sigma_p - \sigma_q} - (S_{22}^{pq} - S_{12}^{pq})\frac{\sigma_q}{\sigma_p - \sigma_q}\right]\right\}^{-1} \tag{4}$$

Note that G_{tc} depends on the principal stresses σ_p and σ_q and on the compliances S^{pq}_{ij}, which in turn depend on the principal stresses. If $\sigma_p = \sigma_q$, then $S^{pq}_{11} = S^{pq}_{22}$, and Eq. (4) reduces to the usual isotropic result:

$$G = [2(S^{pq}_{11} - S^{pq}_{12})]^{-1} \tag{5}$$

Note that, for every case in Eq. (3), the compliances S^{pq}_{ij} in the two-dimensional stress-strain relations are determined on the basis of the signs of the one-dimensional stresses. This situation leads to the requirement

$$\nu_c E_t = \nu_t E_c \tag{6}$$

That is, a reciprocal relation must be satisfied so that the compliance matrix is symmetric (recall the discussion in the preceding section). Ambartsumyan masks some inherent shear coupling in nonprincipal stress coordinates by using just G_{tc}. That is, G_{tc} is not constant in all coordinates but changes as for any orthotropic material. Furthermore, the artificial assignment of the value of cross-compliances (S^{pq}_{12} and S^{pq}_{21}) on the basis of one-dimensional stresses will be shown not to be extendable to the treatment of orthotropic materials.

For an orthotropic body under plane stress, the Ambartsumyan strain-stress relations in principal stress coordinates are[11]

$$\epsilon_p = S^{pq}_{11}\sigma_p + S^{pq}_{12}\sigma_q \tag{7a}$$

$$\epsilon_q = S^{pq}_{21}\sigma_p + S^{pq}_{22}\sigma_q \tag{7b}$$

$$\gamma_{pq} = S^{pq}_{61}\sigma_p + S^{pq}_{62}\sigma_q \tag{7c}$$

where the S^{pq}_{ij} [the compliances in principal stress (p-q) coordinates] take on different values depending on the signs of the principal stresses according to

$$\text{if } \sigma_p > 0 \text{ and } \sigma_q > 0\text{:} \quad S^{pq}_{ij} = S^{pq}_{ijt} \tag{8a}$$

$$\text{if } \sigma_p < 0 \text{ and } \sigma_q < 0\text{:} \quad S^{pq}_{ij} = S^{pq}_{ijc} \tag{8b}$$

$$\text{if } \sigma_p > 0 \text{ and } \sigma_q < 0\text{:} \quad S^{pq}_{11t}, S^{pq}_{12c}, S^{pq}_{21t}, S^{pq}_{22c}, S^{pq}_{61t}, S^{pq}_{62c} \tag{8c}$$

$$\text{if } \sigma_p < 0 \text{ and } \sigma_q > 0\text{:} \quad S^{pq}_{11c}, S^{pq}_{12t}, S^{pq}_{21c}, S^{pq}_{22t}, S^{pq}_{61c}, S^{pq}_{62t} \tag{8d}$$

Thus, an orthotropic multimodulus material is effectively an anisotropic material. The S^{pq}_{ijt} and S^{pq}_{ijc} are related to the S^{mn}_{ijt} and S^{mn}_{ijc} [the compliances in principal material (m-n) coordinates] by the usual transformations of anisotropic elasticity:[14]

$$S^{pq}_{11tc} = S^{mn}_{11tc}\cos^4\beta + (2S^{mn}_{12tc} + S^{mn}_{66tc})\sin^2\beta\cos^2\beta + S^{mn}_{22tc}\sin^4\beta \tag{9a}$$

$$S^{pq}_{12tc} = S^{mn}_{12tc} + (S^{mn}_{11tc} + S^{mn}_{22tc} - 2S^{mn}_{12tc} - S^{mn}_{66tc})\sin^2\beta\cos^2\beta \tag{9b}$$

$$S^{pq}_{21tc} = S^{mn}_{21tc} + (S^{mn}_{11tc} + S^{mn}_{22tc} - 2S^{mn}_{21tc} - S^{mn}_{66tc})\sin^2\beta\cos^2\beta \tag{9c}$$

$$S^{pq}_{22tc} = S^{mn}_{11tc}\sin^4\beta + (2S^{mn}_{12tc} + S^{mn}_{66tc})\sin^2\beta\cos^2\beta + S^{mn}_{22tc}\cos^4\beta \tag{9d}$$

$$S^{pq}_{61tc} = [S^{mn}_{22tc}\sin^2\beta - S^{mn}_{11tc}\cos^2\beta + \tfrac{1}{2}(2S^{mn}_{12tc} + S^{mn}_{66tc})\cos 2\beta]\sin 2\beta \tag{9e}$$

$$S^{pq}_{62tc} = [S^{mn}_{22tc}\cos^2\beta - S^{mn}_{11tc}\sin^2\beta - \tfrac{1}{2}(2S^{mn}_{12tc} + S^{mn}_{66tc})\cos 2\beta]\sin 2\beta \tag{9f}$$

where the subscript tc denotes either t or c as appropriate, β is the angle between principal stress (p-q) directions and principal material (m-n) directions (see Fig. 5), and the S^{mn}_{ijtc} are related to the usual engineering constants by

$$S^{mn}_{11tc} = 1/E_{mmtc} \tag{10a}$$

$$S^{mn}_{12tc} = S^{mn}_{21tc} = -\nu_{mntc}/E_{mmtc} = -\nu_{nmtc}/E_{nntc} \tag{10b}$$

$$S^{mn}_{22tc} = 1/E_{nntc} \tag{10c}$$

$$S^{mn}_{66tc} = 1/G_{mntc} \tag{10d}$$

where $\nu_{mnt} = -\epsilon_n/\epsilon_m$ for $\sigma_m = \sigma_t$ and all other stresses are zero. There are apparently eight independent material properties in Eq. (10), three in tension, three in compression, and two in shear.

Note in Eq. (7) that $\gamma_{pq} \neq 0$; hence, the principal stress directions do not coincide with principal strain directions. Also, Ambartsumyan assigns the cross-compliances S^{pq}_{12} and S^{pq}_{21} on the basis of the signs of σ_p and σ_q, respectively, without requiring that $S^{pq}_{12} = S^{pq}_{21}$. Even if symmetry is enforced in the m-n system, the compliances are not symmetric in any other coordinate system. That is, from the form of the transformation of S^{pq}_{12} in Eq. (9), S_{12t} obviously cannot, in general (i.e., under any rotation whatsoever), be equal to S_{12c} without imposing the overly restrictive and physically improbable relations between the orthotropic material properties suggested by Isabekian and Khachatryan.[16] Thus, the requirement of symmetry of compliances set forth in the preceding section are not met. This point is illustrated in Fig. 6, where the distance from the origin is a measure of the value of the compliance under consideration. (The value changes as the stresses change sign.) The compliance S^{pq}_{11} changes value abruptly as σ_p changes from positive to negative. The compliances S_{12} and S_{21} are equal in the tension-tension and compression-compression quadrants but are not equal in the mixed tension and compression quadrants. For isotropic materials, Ambartsumyan makes the circles coincide by imposing the condition that $\nu_c/E_c = \nu_t/E_t$. However, for orthotropic materials, no such stipulation is made, and so the cross-compliances are not symmetric.

Ambartsumyan[11] and Tabaddor[12] use the Ambartsumyan material model to show that an orthotropic body does exhibit different shear moduli when the sign of the shear stress at 45° to the principal material coordinates is reversed, as in Fig. 4. Note that the shear compliance S^{pq}_{66} is not necessary because it vanishes in all transformations of strains from principal stress coordinates to any other coordinate system.[17]

In summary, the Ambartsumyan material model does not satisfy the criteria for a consistent material model, since the compliances are not symmetric. The principal reason for the asymmetry is that one-dimensional stresses are used to define compliance values in what is actually a two-dimensional stress state. Not all criteria are investigated, since one is violated. However, the apparent shear modulus at 45° to principal material directions for an orthotropic material with different moduli in tension and compression does take on different values under shear stresses of opposite sign. The values of the cross-compliances are assigned more reasonably on the basis

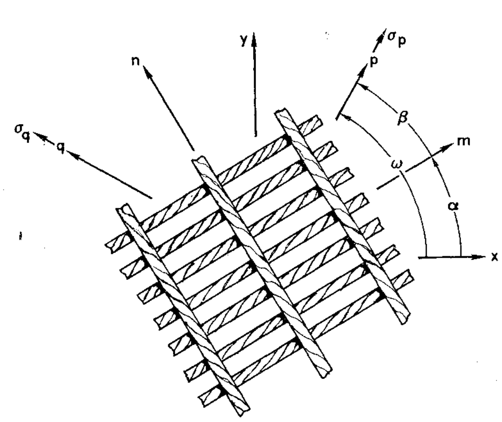

Fig. 5 Relation of material orthotropy to principal stress and body coordinates.

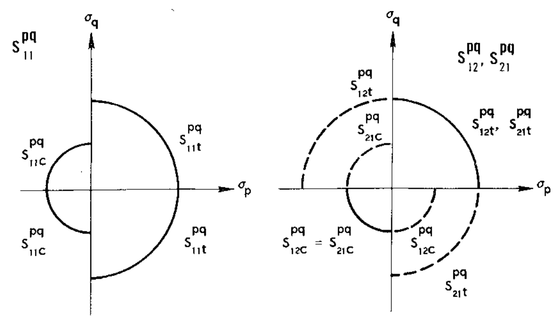

Fig. 6 Compliances in the Ambartsumyan material model.

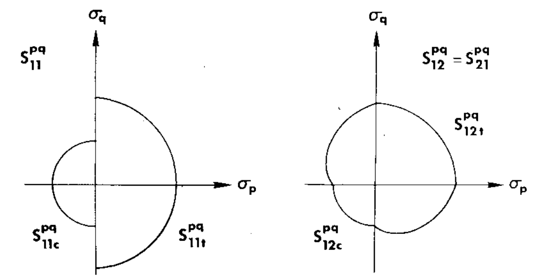

Fig. 7 Compliances in the weighted compliance matrix material model.

of the two-dimensional stress state, and that is the key element of the improved material model proposed in the next section.

Weighted Compliance Matrix Material Model

The object of this section is to present an improved model for materials with different moduli in tension and compression which satisfies the criteria for a consistent material model. First, stress-strain relations for an isotropic material will be discussed, followed by those for an orthotropic material.

For a body under plane stress, the strain-stress relations

$$\epsilon_p = S_{11}^{pq}\sigma_p + S_{12}^{pq}\sigma_q \quad \epsilon_q = S_{12}^{pq}\sigma_p + S_{22}^{pq}\sigma_q \tag{11}$$

in principal stress coordinates are proposed for materials that are isotropic under all-tension or all-compression stress fields. The compliances S_{ij}^{pq} are related to the elastic moduli and Poisson's ratios as follows:

if $\sigma_p > 0$ and $\sigma_q > 0$: $S_{11}^{pq} = 1/E_t, S_{12}^{pq} = -\nu_t/E_t, S_{22}^{pq} = 1/E_t$ (12a)

if $\sigma_p < 0$ and $\sigma_q < 0$: $S_{11}^{pq} = 1/E_c, S_{12}^{pq} = -\nu_c/E_c, S_{22}^{pq} = 1/E_c$ (12b)

if $\sigma_p > 0$ and $\sigma_q < 0$:

$$S_{11}^{pq} = 1/E_t, S_{12}^{pq} = -k_p\nu_t/E_t - k_q\nu_c/E_c, S_{22}^{pq} = 1/E_c \tag{12c}$$

if $\sigma_p < 0$ and $\sigma_q > 0$:

$$S_{11}^{pq} = 1/E_c, S_{12}^{pq} = -k_p\nu_c/E_c - k_q\nu_t/E_t, S_{22}^{pq} = 1/E_t \tag{12d}$$

where

$$k_p = \frac{|\sigma_p|}{|\sigma_p| + |\sigma_q|} \qquad k_q = \frac{|\sigma_q|}{|\sigma_p| + |\sigma_q|} \tag{13}$$

The weighting factors k_p and k_q could be chosen as some other function of the principal stresses. Full qualification of the form of the weighting factors awaits definitive experimental work. At any rate, unlike the Ambartsumyan material model, the compliance matrix is forced to be symmetric without imposing any artificial reciprocal relation between tension and compression properties. The typical resulting variation of compliances with changing principal stresses is illustrated in Fig. 7, wherein the distance from the origin is a measure of the value of the compliance. Note in Fig. 7 how, in contrast to the Ambartsumyan material model, the cross-compliance S_{12}^{pq} always has continuous values as the principal stresses change in proportion to each other. However, the slope of the cross-compliance curve is not continuous. Although difficult to detect in a polar plot, S_{12}^{pq} varies linearly with stress ratio from the all-tension value to the all-compression value. A smoothly varying (i.e., with continuous slope) S_{12}^{pq} would require a weighting function different from that in Eq. (12).

The strain-stress relations in Eq. (11) supplemented by the compliance definitions in Eq. (12) are used to define the improved material model (i.e., an equivalent orthotropic material) for isotropic materials that exhibit different moduli in tension and compression. Recall that S_{66}^{pq} does not appear in the results of strain transformation from principal stress coordinates to any other coordinates. This improved material model includes as a subset the Ambartsumyan model for isotropic materials wherein $\nu_t/E_t = \nu_c/E_c$. Thus, should a material be found that satisfies the rather restrictive Ambartsumyan behavior, that material can be treated directly with the improved material model.

The weighted compliance matrix model obviously satisfies the first criterion of a symmetric compliance matrix, the third criterion of compliance dependence on the biaxial principal stress state, and the fifth criterion of proper reduction to an ordinary isotropic material. The fourth criterion is not applicable to isotropic materials. Investigation of criterion 2 has thus far not been fruitful. For ordinary orthotropic materials, Lempriere[15] derives relationships that are bounds on the behavior of some elastic constants in terms of the values of the other constants. That is, only inequalities are obtained (which, of course, cannot lead to deterministic relations between the elastic constants).

For a body under plane stress, the following strain-stress relations in principal stress (p-q) coordinates are proposed for orthotropic materials that exhibit different moduli in tension and compression:

$$\epsilon_p = S_{11}^{pq}\sigma_p + S_{12}^{pq}\sigma_q \tag{14a}$$

$$\epsilon_q = S_{12}^{pq}\sigma_p + S_{22}^{pq}\sigma_q \tag{14b}$$

$$\gamma_{pq} = S_{61}^{pq}\sigma_p + S_{62}^{pq}\sigma_q \tag{14c}$$

As in the Ambartsumyan model for orthotropic materials, the principal stress directions do not coincide with the principal strain directions. The compliances S_{ij}^{pq} are assigned according to

if $\sigma_p > 0$ and $\sigma_q > 0$: $S_{ij}^{pq} = S_{ijt}^{pq}$ (15a)

if $\sigma_p < 0$ and $\sigma_q < 0$: $S_{ij}^{pq} = S_{ijc}^{pq}$ (15b)

if $\sigma_p > 0$ and $\sigma_q < 0$: $S_{11}^{pq} = S_{11t}^{pq}, S_{12}^{pq} = k_p S_{12t}^{pq} + k_q S_{12c}^{pq}$,

$$S_{22}^{pq} = S_{22c}^{pq}, S_{61}^{pq} = S_{61t}^{pq}, S_{62}^{pq} = S_{62c}^{pq} \tag{15c}$$

if $\sigma_p < 0$ and $\sigma_q > 0$: $S_{11}^{pq} = S_{11c}^{pq}, S_{12}^{pq} = k_p S_{12c}^{pq} + k_q S_{12t}^{pq}$,

$$S_{22}^{pq} = S_{22t}^{pq}, S_{61}^{pq} = S_{61c}^{pq}, S_{62}^{pq} = S_{62t}^{pq} \tag{15d}$$

where k_p and k_q are defined in Eq. (13).

For the cross-compliance S_{12}^{pq}, two stresses must be used in the weighting procedure; e.g., if $\sigma_p > 0$ and $\sigma_q < 0$, then

$$S_{12}^{pq} = k_p S_{12t}^{pq} + k_q S_{12c}^{pq} \tag{16}$$

where both k_p and k_q involve σ_p and σ_q. Even in a three-dimensional stress state, there are at most two stresses that affect the value of a cross-compliance. The modifier "at most" is used because the value of S_{61}^{pq} seems reasonably chosen solely on the basis of the sign of σ_p, i.e.,

$$\text{if } \sigma_p > 0,\ S_{61}^{pq} = S_{61t}^{pq};\ \text{if } \sigma_p < 0,\ S_{61}^{pq} = S_{61c}^{pq} \tag{17}$$

since the other stress, the shear stress, is zero by definition. The cross-compliance S_{62}^{pq} is determined similarly.

The compliances S_{ijt}^{pq} and S_{ijc}^{pq} in the principal stress (p-q) coordinates are related to the compliances S_{ijt}^{mn} and S_{ijc}^{mn} in the principal material (m-n) coordinates by the usual transformation equations of anisotropic elasticity given in Eq. (9). The compliances in principal material coordinates are related to the usual engineering constants by Eq. (10). However, the compliances S_{66t}^{mn} and S_{66c}^{mn} ($1/G_{mnc}$ and $1/G_{mnt}$, respectively) cannot be measured in a shear test on an orthotropic material with different moduli in tension and compression since one principal stress is tension and the other is compression. Instead, in accordance with a suggestion by Tsai,[18] the tension modulus at 45° to the principal material axes, E_t^{45}, is measured, and S_{66t}^{mn} is obtained from

$$S_{66t}^{mn} = \frac{1}{G_{mnt}} = \frac{4}{E_t^{45}} - \left(\frac{1}{E_{mmt}} + \frac{1}{E_{nnt}} - \frac{2\nu_{mnt}}{E_{mmt}}\right) \tag{18}$$

A similar relation is used to define S_{66c}^{mn} when E_c^{45} is known.

Symmetry of the compliance matrix is obtained by assigning $S_{12}^{pq} = S_{21}^{pq}$ on the basis of the biaxial stress state. Thus, the most obvious objections to the Ambartsumyan material model are not present in this improved material model. In principal material coordinates, the anisotropic material is characterized by compliances that are, of course, symmetrical. In any other coordinate system, the matrix of compliances rotated from that given in Eq. (15) is symmetric as well because symmetry is preserved under rotation.

In the body (x-y) coordinates, the S_{ij}^{xy} are obtained from the S_{ij}^{pq} by use of the transformations,[14]

$$S_{11}^{xy} = S_{11}^{pq}\cos^4\omega + (2S_{12}^{pq} + S_{66}^{pq})\sin^2\omega\cos^2\omega$$
$$+ S_{22}^{pq}\sin^4\omega - (S_{61}^{pq}\cos^2\omega + S_{62}^{pq}\sin^2\omega)\sin 2\omega \tag{19a}$$

$$S_{12}^{xy} = S_{12}^{pq} + (S_{11}^{pq} + S_{22}^{pq} - 2S_{12}^{pq} - S_{66}^{pq})\sin^2\omega\cos^2\omega$$
$$- \tfrac{1}{2}(S_{62}^{pq} - S_{61}^{pq})\sin 2\omega\cos 2\omega \tag{19b}$$

$$S_{61}^{xy} = -[S_{22}^{pq}\sin^2\omega - S_{11}^{pq}\cos^2\omega + \tfrac{1}{2}(2S_{12}^{pq} + S_{66}^{pq})\cos 2\omega]\sin 2\omega$$
$$+ S_{61}^{pq}\cos^2\omega(\cos^2\omega - 3\sin^2\omega) + S_{62}^{pq}\sin^2\omega(3\cos^2\omega - \sin^2\omega) \tag{19c}$$

$$S_{22}^{xy} = S_{11}^{pq}\sin^4\omega + (2S_{12}^{pq} + S_{66}^{pq})\sin^2\omega\cos^2\omega$$
$$+ S_{22}^{pq}\cos^4\omega + (S_{61}^{pq}\sin^2\omega + S_{62}^{pq}\cos^2\omega)\sin 2\omega \tag{19d}$$

$$S_{62}^{xy} = -[S_{22}^{pq}\cos^2\omega - S_{11}^{pq}\sin^2\omega - \tfrac{1}{2}(2S_{12}^{pq} + S_{66}^{pq})\cos 2\omega]\sin 2\omega$$
$$+ S_{61}^{pq}\sin^2\omega(3\cos^2\omega - \sin^2\omega) + S_{62}^{pq}\cos^2\omega(\cos^2\omega - 3\sin^2\omega) \tag{19e}$$

$$S_{66}^{xy} = S_{66}^{pq} + 4(S_{11}^{pq} + S_{22}^{pq} - 2S_{12}^{pq} - S_{66}^{pq})\sin^2\omega\cos^2\omega$$
$$- 2(S_{62}^{pq} - S_{61}^{pq})\sin 2\omega\cos 2\omega \tag{19f}$$

The only useful purpose for the S_{ij}^{xy} is to calculate strains that are unaffected by the value of S_{66}^{pq}.[17] Thus, the value of S_{66}^{pq} in Eq. (19) is immaterial and hence arbitrary even though different compliances S_{ij}^{xy} then can be obtained.

The WCM material model can be shown to exhibit different shear moduli when the sign of the shear stress at 45° to principal material directions is reversed. This behavior is apparently in accordance with the known fact that composite material shear strengths are different when the shear stress is reversed.[19] However, part of the reason that the strengths are different must be assumed to be the same reason that the stiffnesses or moduli are different. To demonstrate that the improved material model indeed does exhibit this behavior, consider the cases of "positive" and "negative" shear stress at 45° to principal material directions in Fig. 4. The compliances in the principal stress state for the "positive" shear stress problem are

$$S_{11}^{pq} = S_{11t}^{mn},\ S_{12}^{pq} = \tfrac{1}{2}(S_{12t}^{mn} + S_{12c}^{mn})$$
$$S_{22}^{pq} = S_{22c}^{mn},\ S_{61}^{pq} = 0,\ S_{62}^{pq} = 0 \tag{20}$$

At 45° to the principal stress and principal material coordinates, pure shear stress exists and, from Eq. (19),

$$S_{66}^{45°} = S_{11}^{pq} + S_{22}^{pq} - 2S_{12}^{pq} \tag{21}$$

Note that S_{66}^{pq} identically vanishes from the calculation to get Eq. (21). The shear modulus is the inverse of $S_{66}^{45°}$ and can be expressed in terms of the S_{ij}^{mn} through use of Eq. (20) as

$$G_t^{45°} = (S_{11t}^{mn} + S_{22c}^{mn} - S_{12t}^{mn} - S_{12c}^{mn})^{-1} \tag{22}$$

The compliances in the principal stress state for the "negative" shear stress problem are

$$S_{11}^{pq} = S_{11c}^{mn},\ S_{12}^{pq} = \tfrac{1}{2}(S_{12t}^{mn} + S_{12c}^{mn})$$
$$S_{22}^{pq} = S_{22t}^{mn},\ S_{61}^{pq} = 0,\ S_{62}^{pq} = 0 \tag{23}$$

Thus, upon substitution of Eq. (23) in Eq. (21),

$$G_c^{45°} = (S_{11c}^{mn} + S_{22t}^{mn} - S_{12t}^{mn} - S_{12c}^{mn})^{-1} \tag{24}$$

Finally, upon comparison of $G_t^{45°}$ with $G_c^{45°}$, we see that the WCM material model obviously has different shear moduli when the sign of the shear stress is reversed in coordinates at 45° to principal material directions.

The weighted compliance matrix material model has been demonstrated to satisfy criteria 1 and 3-5 for a consistent material model, but criterion 2 has not led to useful results. That is, the compliance matrix is symmetric, dependent on the biaxial stress state, and reduces properly to single modulus materials. In addition, for orthotropic materials, the apparent shear modulus takes on different values when the sign of the shear stress is reversed.

Comparison of Material Models

The Ambartsumyan and weighted compliance matrix material models are compared for a simple state of plane stress on an isotropic material for which ν_t/E_t is not equal to ν_c/E_c. This condition is treated easily with the WCM model but can only be approximated with the Ambartsumyan model. However, this comparison is fair because real materials generally do not satisfy the Ambartsumyan reciprocal relation, Eq. (6), which is the key element of difference between the two models for isotropic multimodulus materials. An appropriate engineering approximation might be to use the average of the two different cross-compliance values in the Ambartsumyan model. The WCM model is regarded as more accurate since it is more representative of real material behavior.

The schematic variation of S_{12}^{pq} for both models is shown in Fig. 8 as a function of the principal stress state. The approximate Ambartsumyan model is a circle of radius $(\nu_c/E_c + \nu_t/E_t)/2$. The WCM model is a circle of radius ν_t/E_t in the all-tension quadrant, a circle of radius ν_c/E_c in the all-

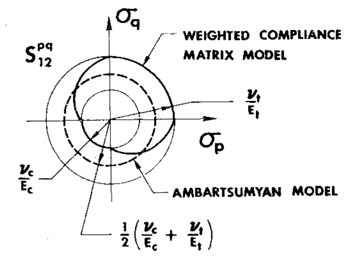

Fig. 8 Comparison of S_{12}^{pq} for Ambartsumyan and weighted compliance matrix models.

compression quadrant, and the usual transition shape in the other two quadrants. The two models conicide only for the two pure shear stress states. The discrepancies are considerable for all other stress states and are largest for all-tension and all-compression stresses.

The effect of the different cross-compliances used in the two models is best assessed by comparing the resulting strains. Consider the simple strain-stress relations of Eq. (11), and subject the body to various principal stress ratios σ_p/σ_q. From Fig. 8, the largest difference between the two cross-compliance approaches occurs in states of uniaxial stress. Thus, consider a state of stress $\sigma_p=\sigma$ and $\sigma_q=0$. The strain-stress relations simplify to

$$\epsilon_p=S_{11}^{pq}\sigma_p, \quad \epsilon_q=S_{12}^{pq}\sigma_p \tag{25}$$

Thus, the only difference between the two models is the predicted value of ϵ_q which is directly proportional to S_{12}^{pq}. Accordingly, we need only examine the average value of S_{12}^{pq} and the weighted value, which in this stress state is S_{12t}, to compare the two models.

Accurate, properly determined mechanical properties for multimodulus materials are scarce. Moreover, the multimodulus effect generally occurs in transversely isotropic or orthotropic materials instead of in simpler isotropic materials. However, in the simple uniaxial stress state of this example, the orthotropy is immaterial. Thus, we examine ATJ-S graphite, a transversely isotropic particulate composite material, under uniaxial load. First, we apply the load in the r direction and then in the z direction of a r-θ-z coordinate system where the r-θ plane is the plane of isotropy of the cylindrical billet. Jortner[20] obtained the properties

$$E_{rt}=1.72\times10^6\text{psi},\ E_{zt}=1.37\times10^6\text{psi},\ \nu_{r\theta t}=0.10,\ \nu_{z\theta t}=0.16$$

$$E_{rc}=1.52\times10^6\text{psi},\ E_{zc}=1.15\times10^6\text{psi},\ \nu_{r\theta c}=0.09,\ \nu_{z\theta c}=0.10$$

For load in the r direction, $S_{12t}=0.10/(1.72\times10^6\text{ psi})=0.0581/10^6$ psi, and $S_{12c}=0.09/(1.52\times10^6\text{ psi})=0.0592/10^6$ psi, whereupon $S_{12\text{av}}=0.05865/10^6$ psi. Thus, the error in using the Ambartsumyan model is -0.95%. For load in the z-direction, $S_{12t}=0.16/(1.37\times10^6\text{ psi})=0.117/10^6$ psi and $S_{12c}=0.10/(1.15\times10^6\text{ psi})=0.0870/10^6$ psi, whereupon $S_{12\text{av}}=0.102/10^6$ psi. Finally, the error in using the Ambartsumyan model is $+12.8\%$. The latter error is clearly unacceptable in most design analyses for ATJ-S graphite applications. The predicted strains with the Ambartsumyan model obviously would be even larger for materials with larger differences between ν_t/E_t and ν_c/E_c.

Concluding Remarks

Composite materials are receiving increasing attention in structural applications because of important potential weight savings. An important characteristic of many composite materials is that they exhibit different moduli or stiffnesses under tensile loading than under compressive loading. In the present paper, mathematical models in the form of stress-strain relations, or constitutive relations, are discussed for materials that have different moduli in tension and compression. Criteria for consistent material models are given which are based on the principles of anisotropic elasticity and the known behavior of materials with different moduli in tension and compression. The Ambartsumyan material model is shown to violate some of the criteria. An improved model, called the weighted compliance matrix material model, is suggested which satisfies the criteria. The new model is described for isotropic and orthotropic bodies under plane stress, but can easily be extended to more complicated situations such as anisotropic bodies under general stress states.

Isotropic materials with different moduli in tension and compression are shown to behave like ordinary orthotropic materials. Moreover, orthotropic materials with different moduli in tension and compression have the characteristics of ordinary anisotropic materials.

An inherent feature of the model, whether Ambartsumyan's or the author's, is that the stress-strain curve is discontinuous at the origin. There, the compliances are indeterminate, but, in a practical situation, either the tension or the compression property would be chosen. For example, at the discontinuity, the lowest modulus (highest compliance) might be chosen. Thus, a prejudice is built into the material model. The discontinuity is evidenced further by the shell buckling results of Jones[13] using the Ambartsumyan material model. Because of the discontinuity, the buckling load changes abruptly upon passage from a biaxial loading involving axial compression and lateral pressure to one involving axial tension and lateral pressure. Such an abrupt change in buckling load would not be expected for a stress-strain curve that is continuous at the origin, e.g., the curve labeled "actual behavior" in Fig. 2. However, the present discontinuous stress-strain curve is the first-order (linear) approximation of the actual nonlinear stress-strain curve.

Definitive experiments have not been performed for materials with different moduli in tension and compression. Simple biaxial mixed tension and compression experiments should be performed on granular composite materials and unidirectionally fiber-reinforced materials, as well as on the more complex multidirectionally woven composite materials (for which, because of their high cost, guidance for experiments by use of analysis results is essential). These experiments are necessary to quantify the concepts discussed in this paper. In particular, the form of the weighting factors used in the improved material model is a somewhat open question. However, the concepts advanced in the present paper should serve as a guide to how to design and conduct experiments. In addition, the model is used in a series of stress analysis and shell buckling papers.[21-23]

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